Perturbation approach to the Feller property for non-local operators (Stochastic Analysis of Jump Processes and Related Topics)

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Perturbation approach to the Feller property for non-local operators

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Abstract

We introduce a perturbation approach studied by the authors to derive the Feller property for a class of integro-differential operators including infinitesimal generators of symmetric stable-like processes. Our approach is based on the stability of the Feller property for infinitesimal generators under bounded perturbations. We also give a simple proof of this fact.

1 Introduction

The main purpose of this article is to give a summary of the paper [21] by the authors which presents a perturbation approach to derive the Feller property for a class of integro-differential operators. Our approach here is applicable to a class of infinitesimal generators of jump-type symmetric Markov processes generated by Dirichlet forms.

Since Fukushima [12, Theorem 4.1] established a one to one correspondence between regular Dirichlet forms and symmetric Hunt processes, the theory of Dirichlet forms has been one of the main tools to construct and study stochastic processes. However, as has already been pointed out in [20], the correspondence is only unique up to an equivalence. Here the equivalence is defined as follows: two symmetric Hunt processes corresponding to a regular Dirichlet form possess a common properly exceptional set outside of which their transition functions coincide ([13, Theorem 4.2.7]). This means that we have some ambiguity concerning starting points in constructing the process. So the question is left whether one can find a nice version from among an equivalence class of stochastic processes which start at each point in a natural way. Some conditions (*e.g.* Feller property or some kind of the strong Feller property) are known in order to construct stochastic processes from every starting point.

Motivated by the fact we mentioned above, we would like to see if the semigroup corresponding to an infinitesimal generator or a (symmetric) Dirichlet form satisfies

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the Feller property. In particular, we are concerned with an integro-differential operator or a jump-type Dirichlet form. One possible way to show the Feller property is to use potential theory. Bass and Levin [6] showed that harmonic functions associated with certain non-local operators are Hölder continuous by using the Harnack inequality and hitting time estimates. This implies the Hölder continuity of the resolvent of the operator. Since then, many authors have studied regularity of harmonic functions associated with non-local operators (see *e.g.* Chen and Kumagai [8, 9], Song and Vondraček [22], Bass and Kassmann [5] and Schilling and Uemura [20]). However, strong conditions are imposed on the jump kernels for the Harnack inequality to hold in those papers. Husseini and Kassmann [15] proved the Feller property of the resolvent corresponding to a jump-type symmetric Dirichlet form under a mild assumption for the kernel without using the Harnack inequality, but using some uniform continuity assumption for the kernel, what they called '*a priori estimate.*' Though we do not use the estimate they assumed, our result is closely related to theirs.

Here we take another approach based on the following stability of the Feller property under a dissipative perturbation. Let X be a locally compact separable metric space and $C_\infty(X)$ the Banach space of continuous functions on X vanishing at infinity equipped with uniform norm. Let A be the infinitesimal generator of a Feller semigroup on $C_\infty(X)$ and B a bounded linear operator on $C_\infty(X)$. If $A+B$ is dissipative, then $A+B$ is again the infinitesimal generator of a Feller semigroup (Proposition 2.1). Such stability of the Feller property has been well-studied (see *e.g.* Davies [10], Pazy [18] and Etheir and Kurtz [11]), and the stability under a dissipative perturbation is already known ([10, Theorem 3.7]). In §2, we also give a simple alternative proof of this fact by making use of the well-known Hille-Yosida theorem and a fixed point theorem.

Using the dissipative perturbation, we show the Feller property for the generators of certain jump-type symmetric Markov processes (Theorem 2.2 and Corollary 3.1). We can then apply this result to a class of symmetric stable-like processes generated by the following regular Dirichlet form:

$$\begin{aligned} \mathcal{E}(u,v) &= \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus D} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha(x)}}\,dxdy \\ \mathcal{F} &= \overline{C_0^{\mathrm{lip}}(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1(\cdot,\cdot)}}. \end{aligned} \tag{1.1}$$

Here $\alpha(x)$ is a positive Borel measurable function on $\mathbb{R}^d$ with suitable conditions, $C_0^{\mathrm{lip}}(\mathbb{R}^d)$ stands for the set of all uniformly Lipschitz functions on $\mathbb{R}^d$ with compact support, $(u,v)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} u(x)v(x)\,dx$ and $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(\mathbb{R}^d)}$, and $D = \{(x,x) \in \mathbb{R}^d \times \mathbb{R}^d\}$. The Dirichlet form of this type is first introduced by the second named author [23] (see also [24]), and such formulation is motivated by the fact that Bass [2] succeeded in constructing a Feller process generated by the non-local operator $(-\Delta)^{\alpha(x)/2}$ $(0<\alpha(x)<2)$, the so-called stable-like process. Furthermore, the L^2-generator of a symmetric stable-like process is obtained from that of a stable-like process by a bounded perturbation. Hence we can construct a symmetric stable-like process which is Feller (see Example 3.3).

We have originally learned the perturbation approach as mentioned above from §4 in [20]. There a given generator is recognized as the "small jumps" part perturbed by the "big jumps" part. They proved that the strong Feller property of the resolvent corresponding to the "small jumps" part implies the Feller property of the generator, while we assume the Feller property of a given operator.

It may seem curious that we derive the Feller property for the generator of a symmetric stable-like process through a (modified) non-symmetric stable-like process. In [25], we revealed a relation between $(-\Delta)^{\alpha(x)/2}$ and the Dirichlet form $(\mathcal{E},\mathcal{F})$ in (1.1) through the *carré du champ operator* associated with $(-\Delta)^{\alpha(x)/2}$. There, we also found that the order of 'the principal (higher order) term' of the difference between the generator of the stable-like process and the L^2-generator of the Dirichlet form $(\mathcal{E},\mathcal{F})$ is the same as that of the stable-like process (see [25, Theorem 3]). This is quite different from the case of diffusion processes. We can not reduce 'the principal term' to the perturbed operator by using a perturbation. Therefore, to show the Feller property for the perturbed operator, we would have to prove the Feller property and solve the martingale problem for it at the same time in general, and this means that it is not necessarily easier to study the symmetric non-local operator than to study the non-symmetric one.

2 Perturbations by bounded linear operators

Let X be a locally compact separable metric space and denote by $C_\infty(X)$ the Banach space of continuous functions on X vanishing at infinity with norm $\|f\|_\infty = \sup_{x\in X}|f(x)|$. By a *Feller semigroup* we mean a strongly continuous, positive contraction semigroup on $C_\infty(X)$. Given a linear operator A on $C_\infty(X)$, we denote its domain by $\mathcal{D}(A)$. We say that the operator A is *dissipative* if

$$\|\lambda u - Au\|_\infty \geq \lambda\|u\|_\infty \text{ for all } u \in \mathcal{D}(A) \text{ and all } \lambda > 0.$$

We now prove the following.

Proposition 2.1. *Let A be the infinitesimal generator of a Feller semigroup on $C_\infty(X)$ and B a bounded linear operator on $C_\infty(X)$. If $A+B$ is dissipative, then $A+B$ is again the infinitesimal generator of a Feller semigroup on $C_\infty(X)$.*

As mentioned in §1, this result is already known as a "dissipative perturbation" (see Theorem 3.7 in [10]), but we give a simple alternate proof for the reader's convenience.

Proof. Let A be the generator of a Feller semigroup on $C_\infty(X)$ and B a bounded linear operator on $C_\infty(X)$. Suppose that $A+B$ is dissipative. We shall then prove that $A+B$ is the generator of a Feller semigroup on $C_\infty(X)$ by applying the Hille-Yosida theorem ([11, p.13, Theorem 2.6]) and a fixed point theorem. To show this, recall the Hille-Yosida theorem: a closed linear operator $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of a Feller semigroup on $C_\infty(X)$ if and only if the following three conditions are satisfied:

(i) $\mathcal{A}$ is dissipative: $\|\lambda u - \mathcal{A}u\|_\infty \geq \lambda\|u\|_\infty$ for all $u \in \mathcal{D}(\mathcal{A})$ and all $\lambda > 0$.

(ii) $\mathcal{D}(\mathcal{A})$ is a dense subset in $C_\infty(X)$.

(iii) the range of $\lambda - \mathcal{A}$ is $C_\infty(X)$ for some $\lambda > 0$.

Note that, by [11, §2.2], the dissipativity implies that (iii) is equivalent to (iii)' the range of $\lambda - \mathcal{A}$ is $C_\infty(X)$ for all $\lambda > 0$.

By assumption, $A + B$ is dissipative. Since $\mathcal{D}(B) = C_\infty(X)$ and $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B) = \mathcal{D}(A)$, we see that $\mathcal{D}(A+B)$ is dense in $C_\infty(X)$. We now show that, for some $\lambda > 0$, the range of $\lambda - (A+B)$ coincides with $C_\infty(X)$. Fix $f \in C_\infty(X)$. For $\alpha > 0$, consider a mapping C_α of $C_\infty(X)$ into itself defined by

$$C_\alpha g = (\alpha - A)^{-1} f + (\alpha - A)^{-1} B g, \quad g \in C_\infty(X).$$

Then it follows from the dissipativity of A that C_α is a contraction mapping for $\alpha > \|B\|$, where $\|B\|$ is the operator norm of B. In fact, for $g, h \in C_\infty(X)$,

$$\begin{aligned}\|C_\alpha g - C_\alpha h\|_\infty &= \left\|(\alpha - A)^{-1}(Bg - Bh)\right\|_\infty \\ &\leq \frac{1}{\alpha}\|Bg - Bh\|_\infty \leq \frac{\|B\|}{\alpha}\|g - h\|_\infty.\end{aligned}$$

Put $\alpha = 2\|B\|$ for instance. Then, by a fixed point theorem, C_α has a unique fixed point $g_0 \in C_\infty(X)$ such that

$$g_0 = C_\alpha g_0 = (\alpha - A)^{-1} f + (\alpha - A)^{-1} B g_0.$$

This equality shows not only $g_0 \in \mathcal{D}(A) = \mathcal{D}(A+B)$ but also

$$f = (\alpha - (A+B)) g_0,$$

that is, the range of $\alpha - (A+B)$ is nothing but $C_\infty(X)$. This completes the proof. □

In what follows, we consider the following integro-differential operator:

$$\mathcal{L}u(x) = \int_{y \neq x} (u(y) - u(x))\, n(x,y) m(dy) \quad \text{for } u \in \mathcal{C}. \tag{2.1}$$

More precisely, let m be a positive Radon measure on X with full support. Let n be a positive measurable function defined on the off-diagonal set $\{(x,y) \in X \times X : x \neq y\}$ such that $\mathcal{L}u \in C_\infty(X)$ for any $u \in \mathcal{C}$, where $\mathcal{C}$ is some dense subset in $C_\infty(X)$. Furthermore, we **assume** that $(\mathcal{L}, \mathcal{C})$ is closable on $C_\infty(X)$ and its closure is the infinitesimal generator of a Feller semigroup on $C_\infty(X)$. For instance, if $k(x,dy) = n(x,y)m(dy)$ is a bounded continuous kernel, then it is easy to see that $\mathcal{L}$ is a bounded linear operator on $C_\infty(X)$ satisfying the Feller property (see *e.g.* [11, §8.3]). Here we say that a kernel $k(x,dy)$ on X is bounded if $k(x,dy)$ is a bounded measure on X for any $x \in X$, and is continuous if $x \mapsto k(x,S)$ is a continuous function on X for any Borel set $S \subset X$. We are also concerned with the following integro-differential operators:

$$\mathcal{A}u(x) = \int_{y \neq x} (u(y) - u(x))\, (n(x,y) + n(y,x)) m(dy)$$

and

$$\tilde{\mathcal{L}}u(x) = \int_{y \neq x} (u(y) - u(x))\, n(y,x) m(dy). \tag{2.2}$$

Here we note that even if $k(x,dy) = n(x,y)m(dy)$ is a bounded kernel, $\tilde{k}(x,dy) := n(y,x)m(dy)$ may be an unbounded kernel. Let B be an integro-differential operator defined by

$$Bu(x) = \int_{y \neq x} (u(y) - u(x))\, (n(y,x) - n(x,y)) m(dy).$$

We now state our main theorem:

Theorem 2.2. *Assume that* $(\mathcal{L},\mathcal{C})$ *given by* (2.1) *is closable on* $C_\infty(X)$ *and its closure is the infinitesimal generator of a Feller semigroup on* $C_\infty(X)$. *Assume further that the operator* B *is a bounded linear operator on* $C_\infty(X)$. *Then* $(\mathcal{A},\mathcal{C})$ *and* $(\tilde{\mathcal{L}},\mathcal{C})$ *are also closable on* $C_\infty(X)$ *and their closures are the infinitesimal generators of Feller semigroups on* $C_\infty(X)$, *respectively.*

Proof. Clearly $\mathcal{A}$ and $\tilde{\mathcal{L}}$ satisfy the positive maximum principle, and thus $\mathcal{A}$ and $\tilde{\mathcal{L}}$ are both dissipative. Now for $u \in \mathcal{C}$, we see that

$$\begin{aligned}\mathcal{A}u(x) &= \int_{y\neq x} (u(y)-u(x))\,(n(x,y)+n(y,x))m(dy)\\ &= 2\int_{y\neq x} (u(y)-u(x))n(x,y)m(dy)\\ &\quad+\int_{y\neq x} (u(y)-u(x))\,(n(y,x)-n(x,y))m(dy)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{L}}u(x) &= \int_{y\neq x} (u(y)-u(x))\,n(y,x)m(dy)\\ &= \int_{y\neq x} (u(y)-u(x))\,n(x,y)m(dy)\\ &\quad+\int_{y\neq x} (u(y)-u(x))\,(n(y,x)-n(x,y))m(dy).\end{aligned}$$

We can then write down $\mathcal{A}$ and $\tilde{\mathcal{L}}$ as $\mathcal{A}=2\mathcal{L}+B$ and $\tilde{\mathcal{L}}=\mathcal{L}+B$, respectively.

Since the closures of $\mathcal{L}$ and $2\mathcal{L}$ on $C_\infty(X)$ are respectively the generators of Feller semigroups and B is a bounded linear operator on $C_\infty(X)$ by assumption, we can extend both operators $\mathcal{A}$ and $\tilde{\mathcal{L}}$ to closed operators on $C_\infty(X)$ that are the infinitesimal generators of Feller semigroups by making use of Proposition 2.1. □

We should note that $\tilde{\mathcal{L}}$ defined in (2.2) is not the 'adjoint operator' of $\mathcal{L}$ on $L^2(X;m)$. In fact, if we denote by $\mathcal{L}^*$ the (formal) adjoint operator of $\mathcal{L}$ on $L^2(X;m)$, then we have the following relation:

$$(\mathcal{L}u,v)_{L^2(X;m)} = (u,\tilde{\mathcal{L}}v)_{L^2(X;m)} + (u,Kv)_{L^2(X;m)}, \quad u,v\in\mathcal{C}\cap L^2(X;m),$$

that is, $\mathcal{L}^*u(x)=\tilde{\mathcal{L}}u(x)+u(x)\cdot K(x)$. Here $(u,v)_{L^2(X;m)}=\int_X u(x)v(x)\,m(dx)$ and

$$K(x)=\int_{y\neq x}\Big(n(y,x)-n(x,y)\Big)m(dy), \quad x\in X.$$

Since the function K may take both positive and negative values, we do not know whether $\mathcal{L}^*$ satisfies the positive maximum principle. Thus $\mathcal{L}^*$ may not be the generator of a sub-Markov semigroup.

3 Applications

There are many jump-type Markov processes on $\mathbb{R}^d$ for which the corresponding generators are the following form:

$$\mathcal{L}u(x)=\int_{y\neq x}\Big(u(y)-u(x)-\nabla u(x)\cdot(y-x)\mathbf{1}_{B(1)}(y-x)\Big)n(x,y)dy, \tag{3.1}$$

where $n(x, y)$ is a positive measurable function defined on the off-diagonal set, $B(r)$ is the closed ball at the origin with radius r and $\mathbf{1}_{B(r)}$ is the indicator function of $B(r)$.

Conversely, if we are given an operator of the type (3.1), then it is natural to ask whether the operator generates a nice Markov process. Related to this problem, many people have considered the martingale problem for operators of this type (see the paper [3] and the books [17] for the references and related topics). However, so far, no one has obtained general conditions on n that guarantee the existence or the uniqueness for the problem. In the sequel, instead of the operator (3.1), we consider the following integro-differential operator:

$$\mathcal{L}u(x) = \int_{y \neq x} (u(y) - u(x))\, n(x, y) dy \quad \text{for } u \in C_0^1(\mathbb{R}^d), \tag{3.2}$$

where $C_0^1(\mathbb{R}^d)$ stands for the set of all continuously differentiable functions on $\mathbb{R}^d$ with compact support. As in §2, let n be a positive measurable function defined on the off-diagonal set such that $\mathcal{L}u \in C_\infty(\mathbb{R}^d)$ for any $u \in C_0^1(\mathbb{R}^d)$. We further assume that $\mathcal{L}$ is closable on $C_\infty(\mathbb{R}^d)$ and its closure is the infinitesimal generator of a Feller semigroup on $C_\infty(\mathbb{R}^d)$. As stated above, if $k(x, dy) = n(x, y)dy$ is a bounded continuous kernel, then $\mathcal{L}$ is a bounded linear operator on $C_\infty(\mathbb{R}^d)$ satisfying the Feller property. On the other hand, if we take $n(x, y) = c(d, \alpha)|x - y|^{-d-\alpha}$ with $0 < \alpha < 1$, then $\mathcal{L}$ is nothing but $-(-\Delta)^{\alpha/2}$ and this generates a symmetric α-stable process on $\mathbb{R}^d$, where $c(d, \alpha)$ is an appropriate constant (see *e.g.* [19, p.217, Example 32.7]).

Corresponding to the operator $\mathcal{L}$, we consider the following carré du champ operator $\Gamma(u, v)$ for $u, v \in C_0^1(\mathbb{R}^d)$ as in [25]:

$$\Gamma(u, v)(x) = \mathcal{L}(u \cdot v)(x) - \mathcal{L}u(x) \cdot v(x) - u(x) \cdot \mathcal{L}v(x), \qquad x \in \mathbb{R}^d.$$

Then it follows from [25, Theorem 1] that

$$\Gamma(u, v)(x) = \int_{y \neq x} (u(x) - u(y))(v(x) - v(y))\, n(x, y) dy.$$

Moreover, we can associate the symmetric quadratic form on $L^2(\mathbb{R}^d)$:

$$\begin{aligned}\mathcal{E}(u, v) &= \int_{\mathbb{R}^d} \Gamma(u, v)(x)\, dx \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus D} \Big(u(x) - u(y)\Big) \Big(v(x) - v(y)\Big) n(x, y) dx dy,\end{aligned}$$

where $D = \{(x, x) \in \mathbb{R}^d \times \mathbb{R}^d\}$. Here it should be emphasized that we can not apply the theory of carré du champ operators developed by [7] and [13] to the operator $\mathcal{L}$ because the kernel $n(x, y)\, dy$ is not necessarily symmetric. In addition, our presentation above gives us a direct relation between the generator and the Dirichlet form.

We now consider whether we can find a Feller process associated to $\mathcal{E}$ from the equivalence class by using the Feller property of $\mathcal{L}$. As for the regularity of the form, we have the following: $\mathcal{D}(\mathcal{E}) = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u, u) < \infty\}$ contains $C_0^1(\mathbb{R}^d)$ if and only if

$$\int_{y \neq x} (1 \wedge |x - y|)\, \nu(x, y) dy \in L^1_{\mathrm{loc}}(\mathbb{R}^d),$$

where $\nu(x,y) = n(x,y) + n(y,x)$ (see [13, Example 1.2.4.], [24]). So, under this condition, $(\mathcal{E}, C_0^1(\mathbb{R}^d))$ is a closable Markovian symmetric form on $L^2(\mathbb{R}^d)$ and there exists a symmetric Hunt process associated with the closure of the form, which is denoted by $(\mathcal{E}, \mathcal{F})$. Next we make a stronger condition on n in order to consider a relation between the generator $\mathcal{L}$ and the Dirichlet form $(\mathcal{E}, \mathcal{F})$. To this end, we denote by $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ the L^2-generator of $(\mathcal{E}, \mathcal{F})$. We assume that

$$\int_{y\neq x} (1 \wedge |x-y|)\, \nu(x,y)dy \in L^2_{\text{loc}}(\mathbb{R}^d) \quad \text{and} \quad \int_{B(r)} \nu(x,y)\, dy \in L^2(\mathbb{R}^d \setminus B(R)) \quad (3.3)$$

for any $r, R > 0$ with $R - r > 1$. It is then showed in [24, Theorem 4.1] that $C_0^1(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{A})$ and

$$\mathcal{A}u(x) = \int_{y\neq x} (u(y) - u(x))\nu(x,y)dy \quad \text{for } u \in C_0^1(\mathbb{R}^d).$$

Let

$$\tilde{\mathcal{L}}u(x) = \int_{y\neq x} (u(y) - u(x))n(y,x)dy \quad \text{for } u \in C_0^1(\mathbb{R}^d) \quad (3.4)$$

and

$$Bu(x) = \int_{y\neq x} (u(y) - u(x))(n(y,x) - n(x,y))\, dy.$$

We can then restate Theorem 2.2 for this setting as follows:

Corollary 3.1. *Assume that $\mathcal{L}$ given by (3.2) is closable on $C_\infty(\mathbb{R}^d)$ and its closure is the infinitesimal generator of a Feller semigroup on $C_\infty(\mathbb{R}^d)$, and that (3.3) holds. Assume further that B is a bounded linear operator on $C_\infty(\mathbb{R}^d)$. Then $\mathcal{A}|_{C_0^1(\mathbb{R}^d)}$ and $\tilde{\mathcal{L}}$ are also closable on $C_\infty(\mathbb{R}^d)$ and their closures are the infinitesimal generators of Feller semigroups on $C_\infty(\mathbb{R}^d)$, respectively.*

Remark 3.2. (i) Fukushima and Stroock [14] discussed a relationship between a Markov process as a solution of the martingale problem and one associated with a (symmetric) Dirichlet form. In particular, their results are applicable to our setting: the closable operator $\mathcal{A}|_{C_0^1(\mathbb{R}^d)}$ on $C_\infty(\mathbb{R}^d)$ and the Dirichlet form $(\mathcal{E}, \mathcal{F})$. So it follows from [14, Theorem 2.9] that a symmetric Hunt process associated with $(\mathcal{E}, \mathcal{F})$ has a version that is Feller if the conditions in Corollary 3.1 are satisfied.

(ii) When the kernel $k(x,dy) = n(x,y)dy$ is not bounded, we do not know general conditions on n so that the closure of $\mathcal{L}$ generates a nice Markov process as mentioned before. However, if we consider the quadratic form defined by

$$\eta(u,v) = -(u, \mathcal{L}v)_{L^2(\mathbb{R}^d)} = -\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus D} u(x)\big(v(y) - v(x)\big)n(x,y)\, dydx$$

for some nice functions u, v, then we will be able to show that this quadratic form becomes a (non-symmetric) regular Dirichlet form under an appropriate condition on n (see [26]).

Here we give a simple but useful remark. Let w be a bounded continuous function defined on $\mathbb{R}^d$ so that for some $\lambda > 0$,

$$\lambda \leq w(x) \leq \lambda^{-1} \quad \text{for all } x \in \mathbb{R}^d,$$

and define

$$\mathcal{L}_w u(x) := w(x)\mathcal{L}u(x) = w(x)\int_{y\neq x}(u(y)-u(x))n(x,y)dy \quad \text{for } u\in C_0^1(\mathbb{R}^d),$$

where $\mathcal{L}$ is the operator given by (3.2). If all the conditions in Corollary 3.1 are satisfied, then $\mathcal{L}_w$ satisfies the positive maximum principle, and hence is dissipative. So, we can conclude that $\mathcal{L}_w$ is also the generator of a Feller semigroup. In particular, the Feller process associated with $\mathcal{L}_w$ is a time changed process of that associated with $\mathcal{L}$.

Example 3.3. (stable-like process). Let $\alpha(x)$ be a positive measurable function on $\mathbb{R}^d$. Define for $u\in C_0^2(\mathbb{R}^d)$,

$$\begin{aligned}\mathcal{L}_w u(x) &:= w(\alpha(x))\mathcal{L}u(x)\\ &:= w(\alpha(x))\int_{h\neq 0}\frac{u(x+h)-u(x)-\nabla u(x)\cdot h\mathbf{1}_{\{|h|\leq 1\}}(h)}{|h|^{d+\alpha(x)}}dh,\end{aligned}$$

where

$$w(\alpha)=\frac{\Gamma(1+\alpha/2)\Gamma((\alpha+d)/2)\sin((\pi\alpha)/2)}{2^{1-\alpha}\pi^{d/2+1}}. \tag{3.5}$$

Then $\mathcal{L}_w e^{iu\cdot x} = -|u|^{\alpha(x)}e^{iu\cdot x}$ holds (see e.g. [1, p.402]).

Assume that

$$0<\inf_{x\in\mathbb{R}^d}\alpha(x)\leq\sup_{x\in\mathbb{R}^d}\alpha(x)<2, \tag{3.6}$$

and that for some $M>0$,

$$|\alpha(x)-\alpha(y)|\leq M|x-y| \quad \text{for } x,y\in\mathbb{R}^d. \tag{3.7}$$

Bass [2] then showed that there exists a unique strong Markov and Feller process $\mathbf{M}=(X_t,P_x)$ such that P_x solves the martingale problem for $\mathcal{L}_w$ at every starting point $x\in\mathbb{R}^d$. Here we can see from (3.5) that if α satisfies the conditions (3.6) and (3.7), then $w(\alpha(\cdot))$ is a bounded continuous function and is also bounded below by some positive constant. Hence $\mathcal{L}=(1/w(\alpha(\cdot)))\mathcal{L}_w$ is also the generator of a Feller semigroup. So, in the sequel we consider the operator $\mathcal{L}$ for simplicity. We now keep (3.7) and assume (3.8) instead of (3.6):

$$0<\inf_{x\in\mathbb{R}^d}\alpha(x)\leq\sup_{x\in\mathbb{R}^d}\alpha(x)<1. \tag{3.8}$$

Then for $u\in C_0^1(\mathbb{R}^d)$, we can reduce the form of $\mathcal{L}u$ as follows:

$$\begin{aligned}\mathcal{L}u(x) &= \int_{h\neq 0}\Big(u(x+h)-u(x)\Big)|h|^{-d-\alpha(x)}\,dh\\ &= \int_{y\neq x}\Big(u(y)-u(x)\Big)|x-y|^{-d-\alpha(x)}\,dy.\end{aligned}$$

Namely, this is the case $n(x,y)=|x-y|^{-d-\alpha(x)}$ in (3.2). Note that (3.8) implies (3.3) as mentioned in [24, Example 4.1].

We now show that the operator B defined by

$$Bu(x)=\int_{h\neq 0}(u(x+h)-u(x))\left(|h|^{-d-\alpha(x+h)}-|h|^{-d-\alpha(x)}\right)dh \tag{3.9}$$

is a bounded linear operator on $C_\infty(\mathbb{R}^d)$ directly. In fact, we first see that

$$\begin{aligned}\left||h|^{-\alpha(x)} - |h|^{-\alpha(x+h)}\right| &= \left|\int_{\alpha(x+h)}^{\alpha(x)} |h|^{-u} \log\frac{1}{|h|}\,du\right| \\ &\leq |\alpha(x+h) - \alpha(x)||h|^{-(\alpha(x)\vee\alpha(x+h))} \log\frac{1}{|h|} \\ &\leq M|h|^{1-\overline{\alpha}} \log\frac{1}{|h|} \quad \text{for } 0 < |h| < 1\end{aligned}$$

in a similar way to [20, Example 2.6] (see also the proof of [16, Lemma 3.1]) and that $\left||h|^{-\alpha(x)} - |h|^{-\alpha(x+h)}\right| \leq 2|h|^{-\underline{\alpha}}$ for $|h| > 1$, where

$$0 < \underline{\alpha} := \inf_{x\in\mathbb{R}^d} \alpha(x) \leq \overline{\alpha} := \sup_{x\in\mathbb{R}^d} \alpha(x) < 1.$$

So the absolute value of the integrand in the right hand side of (3.9) is dominated by

$$c\|u\|_\infty\left(\mathbf{1}_{\{|h|\leq 1\}} \cdot |h|^{1-d-\overline{\alpha}} \log(1/|h|) + \mathbf{1}_{\{|h|>1\}} \cdot |h|^{-d-\underline{\alpha}}\right)$$

for some constant $c > 0$ independent to x and h, which is integrable on $\{h \neq 0\}$ with respect to the Lebesgue measure. Consequently, by virtue of the dominated convergence theorem, we find that B is a bounded linear operator on $C_\infty(\mathbb{R}^d)$.

Let

$$\mathcal{E}(u,v) = \iint_{h\neq 0} \frac{(u(x+h) - u(x))(v(x+h) - v(x))}{|h|^{d+\alpha(x)}}\,dhdx \quad \text{for } u, v \in C_0^1(\mathbb{R}^d) \tag{3.10}$$

and $\mathcal{F} = \overline{C_0^1(\mathbb{R}^d)}^{\sqrt{\mathcal{E}_1(\cdot,\cdot)}}$. Then $(\mathcal{E},\mathcal{F})$ generates a symmetric stable-like process introduced in [23]. Moreover, if we denote by $\mathcal{A}$ the $L^2(\mathbb{R}^d)$-generator of $(\mathcal{E},\mathcal{F})$, then Corollary 3.1 shows that

$$\mathcal{A}|_{C_0^1(\mathbb{R}^d)}u(x) = \int_{h\neq 0} (u(x+h) - u(x))\left(|h|^{-d-\alpha(x)} + |h|^{-d-\alpha(x+h)}\right) dh$$

is closable on $C_\infty(\mathbb{R}^d)$ and its closure generates a Feller semigroup on $C_\infty(\mathbb{R}^d)$. Namely, the corresponding symmetric stable-like process can be refined to be a Feller process. We see that

$$\tilde{\mathcal{L}}u(x) = \int_{h\neq 0} (u(x+h) - u(x))|h|^{-d-\alpha(x+h)}dh \quad \text{for } u \in C_0^1(\mathbb{R}^d)$$

is also closable on $C_\infty(\mathbb{R}^d)$ and its closure generates a Feller process.

Finally, we consider the following operator:

$$\mathcal{L}_1u(x) = \int_{h\neq 0} \left(u(x+h) - u(x)\right)\frac{c(x,h)}{|h|^{d+\alpha(x)}}dh \quad \text{for } u \in C_0^1(\mathbb{R}^d),$$

where $c(x,h)$ is a bounded, nonnegative and continuous function defined on $\mathbb{R}^d\times\mathbb{R}^d$. We assume that there exist positive constants $M > 0$ and $\delta > 0$ with $\overline{\alpha} < \delta \leq 1$ such that

$$|c(x,h) - 1| \leq M|h|^\delta \quad \text{for } x \in \mathbb{R}^d \text{ and } |h| \leq 1.$$

We can then write down $\mathcal{L}_1$ as follows:

$$\begin{aligned}\mathcal{L}_1 u(x) &= \int_{h\neq 0} \Big(u(x+h) - u(x)\Big) \frac{1}{|h|^{d+\alpha(x)}} dh \\ &+ \int_{h\neq 0} \Big(u(x+h) - u(x)\Big) \frac{c(x,h)-1}{|h|^{d+\alpha(x)}} dh \\ &=: \mathcal{L}u(x) + B_1 u(x) \quad \text{for } x \in \mathbb{R}^d.\end{aligned}$$

Using the assumption for $c(x,h)$, we can easily show that B_1 is a bounded linear operator on $C_\infty(\mathbb{R}^d)$. Therefore, we see from Proposition 2.1 that $\mathcal{L}_1$ is closable and its closure generates a Feller semigroup on $C_\infty(\mathbb{R}^d)$ (*c.f.* Theorem 5.1 and 5.2 in [4]).

Let

$$\mathcal{G}(u,v) = \iint_{h\neq 0} (u(x+h) - u(x))(v(x+h) - v(x)) \frac{c(x,h)}{|h|^{d+\alpha(x)}} dhdx \qquad \text{for } u, v \in C_0^1(\mathbb{R}^d)$$

and $\mathcal{H} = \overline{C_0^1(\mathbb{R}^d)}^{\sqrt{\mathcal{G}_1(\cdot,\cdot)}}$. Then $(\mathcal{G},\mathcal{H})$ is a regular Dirichlet form and generates a jump-type symmetric Markov process. Let $(\mathcal{A}_1, \mathcal{D}(\mathcal{A}_1))$ be the $L^2(\mathbb{R}^d)$-generator of $(\mathcal{G},\mathcal{H})$. Then

$$\mathcal{A}_1|_{C_0^1(\mathbb{R}^d)} u(x) = \int_{h\neq 0} \Big(u(x+h) - u(x)\Big) \left(\frac{c(x,h)}{|h|^{d+\alpha(x)}} + \frac{c(x+h,-h)}{|h|^{d+\alpha(x+h)}}\right) dh.$$

Let us define the operator $\tilde{B}_1$ by

$$\tilde{B}_1 u(x) = \int_{h\neq 0} \Big(u(x+h) - u(x)\Big) \frac{c(x+h,-h)-1}{|h|^{d+\alpha(x+h)}} dh.$$

Then $\tilde{B}_1$ is also a bounded linear operator on $C_\infty(\mathbb{R}^d)$. Since

$$\mathcal{A}_1|_{C_0^1(\mathbb{R}^d)} u(x) = \mathcal{A}|_{C_0^1(\mathbb{R}^d)} u(x) + B_1 u(x) + \tilde{B}_1 u(x) \quad \text{for } x \in \mathbb{R}^d,$$

Proposition 2.1 implies that $\mathcal{A}_1|_{C_0^1(\mathbb{R}^d)}$ is closable on $C_\infty(\mathbb{R}^d)$ and its closure generates a Feller semigroup on $C_\infty(\mathbb{R}^d)$. Hence a Markov process corresponding to $(\mathcal{G},\mathcal{H})$ can be refined to be a Feller process. Set

$$\tilde{\mathcal{L}}_1 u(x) = \int_{h\neq 0} \Big(u(x+h) - u(x)\Big) \frac{c(x+h,-h)}{|h|^{d+\alpha(x+h)}} dh \quad \text{for } u \in C_0^1(\mathbb{R}^d).$$

By noting that

$$\tilde{\mathcal{L}}_1 u(x) = \tilde{\mathcal{L}} u(x) + \tilde{B}_1 u(x) \quad \text{for } x \in \mathbb{R}^d$$

for $u \in C_0^1(\mathbb{R}^d)$, the operator $\tilde{\mathcal{L}}_1$ is also closable on $C_\infty(\mathbb{R}^d)$ and its closure generates a Feller semigroup on $C_\infty(\mathbb{R}^d)$.

Remark 3.4. Because of the restricted conditions on n, we can only show the Feller property of the processes corresponding to the form (3.10) under (3.7) and (3.8). So, the general case (3.6) still remains open.

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